A PROOF OF THE GENERALIZED SECOND-LIMIT THEOREM IN THE THEORY OF PROBABILITY*

BY

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Introduction. A function $F(x)$, defined for all real x, will be called a "law of probability," if the following conditions are satisfied:
(i) $F(x)$ is monotone non-decreasing in $(-\infty, \infty)$ and continuous to the left,

(ii) $$F(-\infty) = 0, \ F(\infty) = 1.\dagger$$

A particular case is represented by $dF(x) = f(x)dx$, where $f(x)$, summable and $\geqq 0$, is the "probability density" or "law of distribution" for x.

The expression $\int_{-\infty}^{\infty} x^s dF(x)$ is called the "sth moment" of the distribution, s taking values $0, 1, 2, \cdots$.

The Second Limit-Theorem, which was the starting point of this paper, can be stated, with A. Markoff,‡ as follows:

If a sequence of laws of probability $F_k(x)$ $(k=1, 2, \cdots)$ is such that they admit moments of all orders, and if

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} x^s dF_k(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^s e^{-x^2} dx \qquad (s = 0, 1, \cdots),$$

then, for all x,

$$\lim_{k\to\infty} \int_{-\infty}^{x} dF_k(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-x^2} dx.$$

Markoff's proof is rather complicated, being based on the distribution of roots and other properties of Hermite polynomials, also on the so-called Tchebycheff inequalities in the theory of algebraic continued fractions. He points out that the theorem still holds if we replace the law of probability $\pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$ by a more general one: $\int_{-\infty}^{x} f(x)dx$ (in which case, however, his considerations need many supplements).§

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† In fact, if X is a fortuitous variable (finite, not necessarily bounded), and if $F(x)$ is the probability that $X<x$, then $F(x)$ will satisfy these conditions, provided we assume that the principle of total probabilities still holds for a *countable infinity* of inconsistent events.

‡ A. Markoff, *Theory of Probability*, 4th edition (1924, in Russian), p. 522.

§ Cf. J. Chokhate, *Sur la convergence des quadratures mécaniques dans un intervalle infini* . . . , Comptes Rendus, vol. 186 (1928), pp. 344–346.

The same theorem has recently attracted the attention of many investigators: R. von Mises,* G. Pólya,† Paul Lévy,‡ Cantelli,§ Jacob|| and others.

The object of this paper is (a) to establish a general limit-theorem, removing many restrictions imposed otherwise on the functions involved and their moments, so that the above statement dealing with the law of probability $\pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$ (we shall call it hereafter the "classical case") is therein included as a very special case; (b) to give an *elementary* proof, which does not use either characteristic functions or algebraic continued fractions, being based on a well known Montel-Helly theorem concerning sequences of monotonic functions.

A brief account will first be given of the "moments-problem" to which the theorem in question is closely related.

1. **The moments-problem.** *Given a certain interval (a, b), finite or infinite, and an infinite sequence of real constants $c_0, c_1, \cdots$, find a function $\psi(x)$, nondecreasing in (a, b),¶ such that*

$$\int_a^b x^s d\psi(x) = c_s \qquad (s = 0, 1, \cdots).$$

We call this the moments-problem corresponding to the data $\{c_s\}$. We may assume, without loss of generality, $\psi(a) = 0$.

It is known that *if (a, b) be finite, then the moments-problem cannot have more than one solution* (if it has any),** if we generally agree to *consider as*

* R. von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1–97.

† G. Pólya, *Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung ...*, Mathematische Zeitschrift, vol. 8 (1920), pp. 171–181.

‡ Paul Lévy, *Calcul des Probabilités*, Paris, 1925, Chapter IV.

§ F. P. Cantelli, *Una nuova dimostrazione del secondo teorema-limite ...*, Rendiconti di Palermo, vol. 52 (1928), pp. 151–174.

|| Jacob, *De l'application des intégrales généralisées de Fourier au calcul des probabilités*, Comptes Rendus, vol. 188 (1929), pp. 541–43, 754–56.

¶ The case of $\psi(x)$ having but a finite number of points of increase in (a, b) is trivial.

** A simple proof is the following. The existence of two solutions $\psi_1(x), \psi_2(x)$ implies

$$\int_a^b x^s dF(x) = 0 \qquad (s = 0, 1, \cdots; F(x) = \psi_1 - \psi_2; F(a) = 0),$$

$$F(b) = F(a) \quad (\text{for } s = 0), \quad \int_a^b x^l F(x)dx = 0 \quad (l = 0, 1, \cdots; \text{integration by parts}).$$

The latter relations lead to the required conclusion: $F(x) = 0$ at all points of continuity in (a, b), by the following reasoning due to Stieltjes (*Correspondance d'Hermite et Stieltjes*, Paris, 1905, pp. 337–338). If such a point z exists $(a < z < b)$, where $F(z) \neq 0$, then $1-(x-z)^2/M > 0$ $(a \leqq x \leqq b)$, for a sufficiently large M; hence, it is easily seen that

$$I = \int_a^b F(x)[1 - (x-z)^2/M]^n dx,$$

identical two solutions $\psi_1(x)$, $\psi_2(x)$ *which coincide at all points of continuity.** We express this property by saying that *the moments-problem for a finite interval is "determined."*

On the other hand, *the moments-problem for an infinite interval may be "indeterminate,"* i.e., *it may admit infinitely many solutions.* In fact, in the formula

$$\int_0^\infty y^{a-1}e^{-by}dy = \frac{\Gamma(a)}{b^a}$$

take

$$b = k + di \quad (k > 0), \; a = (n+1)/\lambda, \; (2n+1)/\lambda \quad (n = 0, 1, \cdots),$$

$$\frac{d}{k} = \tan \lambda\pi, \quad \tan\frac{\mu\pi}{2} \quad (\lambda, \mu \text{ defined below}), \quad y = x^\lambda,$$

and we get functions *having all moments* $=0$:†

$$\int_0^\infty x^n e^{-\kappa x^\lambda} \sin(\kappa x^\lambda \tan \lambda\pi) dx = 0 \qquad (\kappa > 0, 0 < \lambda < \tfrac{1}{2}),$$

$$\int_{-\infty}^\infty x^n e^{-\kappa x^\mu} \cos\left(\kappa x^\mu \tan\frac{\mu\pi}{2}\right) dx = 0$$

$$\left(\kappa > 0, 0 < \mu = \frac{2s}{2s+1} < 1, s \text{ a positive integer}\right).$$

Hence we get infinitely many non-decreasing functions

$$(1) \qquad \begin{aligned} &\int_0^x e^{-\kappa x^\lambda}[1 + h \sin(\kappa x^\lambda \tan \lambda\pi)]dx, \\ &\int_{-\infty}^x e^{-\kappa x^\mu}\left[1 + h\cos\left(\kappa x^\mu \tan\frac{\mu\pi}{2}\right)\right]dx \qquad (-1 \leqq h \leqq 1), \end{aligned}$$

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for n very large, is different from zero, *which is impossible,* I being a linear combination of the moments of $F(x)$, all of which vanish. (We notice that such M does not exist for (a, b) infinite.) Moreover, if $\psi_i(x)$ is continuous to the left, then $\psi_1(x) = \psi_2(x)$ *everywhere in* (a, b), since $\psi_i(x-0) = \lim \psi_i(X)$, where $X(<x)$ converges to x, being always a point of continuity of $\psi_i(x)$, $i = 1, 2$.

* Also at the points a, b, if (a, b) be finite; this, however, necessarily follows from the relations

$$\psi_{1,2}(a) = 0, \quad \int_a^b d\psi_1(x) = \int_a^b d\psi_2(x) = c_0.$$

† These have been given by Adamoff (*Proof of a theorem of Stieltjes,* Proceedings of the Kazan Mathematical Society (1911, in Russian)) and by Stekloff (*Application de la théorie de fermeture . . . ,* Mémoires de l'Académie des Sciences, Petrograd, vol. 33 (1915)), but the original statement is due to Stieltjes (loc. cit., p. 230).

solutions of the same moments-problem for $(0, \infty)$ and $(-\infty, \infty)$ respectively. Either of the following conditions ensures the determined character of the moments-problem for an infinite interval:

$$(2) \qquad \sum_{n=1}^{\infty} c_{2n}^{-1/(2n)} \text{ diverges*} \qquad \left(c_n = \int_{-\infty}^{\infty} x^n dF(x)\right);$$

$$(3) \qquad dF(x) = p(x)dx \quad (p(x) \geqq 0 \text{ on } (a, b))$$

with $p(x) < M|x|^{\alpha-1}e^{-\kappa|x|^\lambda}$ for $|x| \geqq x_0$, sufficiently large (M, α, κ are positive constants); $\lambda \geqq \frac{1}{2}$ for $(a, b) = (0, \infty)$,† $\lambda \geqq 1$ for $(a, b) = (-\infty, \infty)$.‡

On the other hand, the moments-problem is indeterminate if $d\psi(x) = p(x)dx$, and for $|x|$ sufficiently large (see (1))

$$p(x) > e^{-\kappa|x|^\lambda} \ (\kappa > 0) \text{ with } \lambda < \tfrac{1}{2} \text{ for } (0, \infty), \ \lambda < 1 \text{ for } (-\infty, \infty).$$

2. **A generalized statement of the second limit-theorem.** *Given a sequence of laws of probability* $F_n(x)$ $(n = 1, 2, \cdots)$, *with the following properties*: (i) *the moments* $m_n^{(r)} = \int_{-\infty}^{\infty} x^r dF_n(x)$ *of all orders* $r = 0, 1, \cdots$ *exist for* $n = 1, 2, \cdots$, *or at least from a certain rank n on (possibly depending on r)*; (ii) *the quantities* $m_1^{(r)}, m_2^{(r)}, \cdots$, *for any* $r = 0, 1, \cdots$, *lie, when they exist, between two fixed limits independent of n (but possibly dependent on r). Then a subsequence* $\{C_i(x) \equiv F_{n_i}(x)\}$ $(i = 1, 2, \cdots; n_1 < n_2 < \cdots; n_i \to \infty)$ *can be extracted such that* (α) $\lim_{i\to\infty} \int_{-\infty}^{\infty} x^r dC_i(x)$ *exists* $(= m_r)$,§ *for* $r = 0, 1, \cdots$; (β) *the subsequence* $\{C_i(x)\}$ *converges for any x to one fixed law of probability* $\psi(x)$, *save, perhaps, at its points of discontinuity*; (γ) $\int_{-\infty}^{\infty} x^r d\psi(x)$ *exists and* $= m_r (r = 0, 1, \cdots)$.‖

The proof will be arranged in several steps.

3. **Existence of** $m_r (r = 1, 2, \cdots)$. We apply here the classical "diagonal method." The hypothesis of the uniform boundedness of $\{m_n^{(r)}\}$ for all $r = 1, 2, \cdots$, enables us to extract from the sequence $\{m_n^{(1)}\}$ a subsequence $\{m_{p_i}^{(1)}\}$ converging to a finite limit m_1. The sequence $\{m_{p_i}^{(2)}\}$ gives rise to a subsequence $\{m_{q_i}^{(2)}\}$ converging to a finite limit m_2, and so on. We thus get a sequence $\{m_{p_1}^{(r)}, m_{q_2}^{(r)}, \cdots\}$ converging to a finite limit m_r, for any

* T. Carleman, *Sur les équations intégrales singulières* . . . , Uppsala, 1923, p. 219.

† Stieltjes, *Recherches sur les fractions continues*, Oeuvres, vol. II, pp. 402–559, where (3) is given for $(0, \infty)$ only. The corresponding condition for $(-\infty, \infty)$ follows directly.

‡ It can be easily shown that, if $x_0 > 1$ and if, in $(1, x_0)$, $p(x)$ is bounded in the sense of Lebesgue, i.e., disregarding a set of zero measure, then (3) is included in (2). Furthermore, if $\psi_1(x), \psi_2(x)$ are solutions of a determined moments-problem for the interval (a, ∞) $(a = 0, -\infty)$, we can arrange so as to have $\psi_1(x) = \psi_2(x)$ *everywhere in* (a, ∞), since $\psi_1(x) = \psi_2(x)$ at all points of continuity, it being permissible, as above, to take $\psi_i(x)$ continuous to the left $(i = 1, 2)$.

§ m_0 exists and $= 1$, by definition of law of probability.

‖ It follows that there is necessarily at least one solution of the moments-problem corresponding to the data $\{m_r\}$.

$r=1, 2, \cdots$. The corresponding laws of probability $\gamma_1(x)\equiv F_{p_1}(x)$, $\gamma_2(x)\equiv F_{q_2}(x), \cdots$ clearly satisfy (α). The reasoning still holds if none of the $F_n(x)$ has *all* of its moments finite.

4. **Existence of a limiting law of probability $\psi(x)$.** This follows by applying to $\{F_n(x)\}$ the Montel-Helly* theorem on monotonic functions. We state it in a slightly generalized form:

If a family $\{f(x)\}$ of functions, non-decreasing on $(-\infty, \infty)$, is uniformly bounded in any finite interval (i.e. $|f(x_0)| < A(x_0)$ at any finite point x_0, $A(x_0)$ being the same for all $f(x)$), then from any infinite sequence of this family we can extract a subsequence which converges, for any x, to a non-decreasing function. Moreover, the convergence is uniform in any interval, where the limit-function is continuous.

The theorem holds, with proper modifications, for families of functions of bounded variation.

In order to prove (β), it suffices to apply this theorem to the sequence $\{\gamma_i(x)\}$, since $0\leqq\gamma_i(x)\leqq 1$ for $-\infty\leqq x\leqq\infty$. We extract then from it a sequence $\{C_p(x)\}$ converging everywhere to a non-decreasing function $\phi(x)$, and we take $\psi(x)\equiv\phi(x-0)$.

The following remarks are important:

(i) The limit-function $\psi(x)$ of the $\{C_p(x)\}$ varies *effectively* from 0 to 1, i.e.

$$\psi(-\infty) = 0, \ \psi(\infty) = \int_{-\infty}^{\infty} d\psi(x) = 1.\dagger$$

(ii) For any $r=0, 1, \cdots$, the convergence of

$$\lim_{a\to-\infty,\, b\to\infty} \int_a^b x^r dF_n(x) \text{ to } \int_{-\infty}^{\infty} x^r dF_n(x)$$

is uniform with respect to n, at least from a certain rank n on.

(iii) $\lim_{x\to\infty} x^s(1 - F_n(x)) = 0, \ \lim_{x\to-\infty} |x|^s F_n(x) = 0 \ (n = 1,2, \cdots; s > 0 \text{ arbitrary}).$

(i) follows from the inequalities

$$(4) \qquad \int_b^{\infty} x^r dF_n(x) \leqq \frac{\overset{(2r+2)}{m_n}}{b^{r+2}} \quad (b > 1); \ \int_{-\infty}^{a} x^r dF_n(x) \leqq \frac{\overset{(2r+2)}{m_n}}{|a|^{r+2}} \quad (a < -1),$$

* (a) P. Montel, *Sur les suites infinies des fonctions*, Annales de l'École Normale Supérieure, 1907; (b) E. Helly, *Über lineare Funktionaloperationen*, Sitzungsberichte der Akademie der Wissenschaften, Wien, vol. 121 (1912), pp. 265–297.

† This is by no means obvious. Take, for example, $F_n(x)=0\ (x<-n)$, $=\frac{1}{2}(-n\leqq x\leqq n)$, $=1(x>n)$. Here $\lim_{n\to\infty}F_n(x)=\psi(x)\equiv\frac{1}{2}$ for $-\infty\leqq x\leqq\infty$.

which, applied to $\{C_p(x)\}$, yield, for $r=0$ and $p\to\infty$,

$$(5) \qquad (0 \leqq) 1 - \psi(b) \leqq \frac{1+m_2}{b^2}, \; 0 \leqq \psi(a) \leqq \frac{1+m_2}{a^2} \qquad (b, -a > 1),$$

and this proves (i) by letting $b\to\infty$, $a\to-\infty$.

(ii) follows directly from (4), taking into account the uniform boundedness of $\{m_n^{(2r+2)}\}$ $(n=1, 2, \cdots)$.

In order to establish (iii), we write

$$1 - F_n(b) = \int_b^\infty dF_n(x) \leqq \int_b^\infty \left(\frac{x}{b}\right)^{2r} dF_n(x); \; b^s[1 - F_n(b)] \leqq \frac{m_n^{(2r)}}{b^{2r-s}}$$

$$(b > 0, 0 < s < 2r),$$

and a similar expression for $|a|^s F_n(a) (a < -1)$.

5. $\int_{-\infty}^{\infty} x^r d\psi(x)$ exists and $= m_r (r=0, 1, \cdots)$. This statement being the fundamental part of the theorem, we give for it two proofs.

Proof I. We apply the two following theorems due respectively to Hamburger* and to Helly,† the proofs of both of which are very elementary.

Hamburger's Theorem. *Suppose* (i) $\int_{-\infty}^{\infty} [1/(z+x)] d\psi(x)$ *converges for* $z=iy$ *with* $y>0$, $\psi(x)$ *denoting a function non-decreasing in* $(-\infty, \infty)$; (ii) $F(z) \equiv \int_{-\infty}^{\infty} [1/(z+x)] d\psi(x)$ *has, for* $z=iy\to\infty$, *an asymptotic expansion* (*in Poincaré's sense*) $F(z) \sim \sum_{\nu=1}^{\infty} (-1)^\nu c_\nu / z^{\nu+1}$ (c_ν *real*). *Then* $\int_{-\infty}^{\infty} x^\nu d\psi(x)$ *exists and* $= c_\nu$ $(\nu=0, 1, \cdots)$.

Helly's Theorem. *Given a sequence* $\{V_n(x)\}$ *of functions of bounded variation on a finite interval* (a, b) *such that* (i) *the total variations of all* $V_n(x)$ *on* (a, b) *are uniformly bounded*; (ii) $\lim_{n\to\infty} V_n(x) = v(x)$‡ *exists for* $a \leqq x \leqq b$, *with the possible exception of a countable set of points not including* a, b. *Then* $\lim_{n\to\infty} \int_a^b f(x) dV_n(x) = \int_a^b f(x) dv(x)$, *for any continuous function* $f(x)$.

Going back to the given sequence $\{F_n(x)\}$ and to the above function $\psi(x) = \lim_{p\to\infty} \{C_p(x)\}$, we notice, first, that $\int_{-\infty}^{\infty} f(x) d\psi(x)$, $\int_{-\infty}^{\infty} f(x) dF_n(x)$ certainly exist if $f(x)$ is bounded on $(-\infty, \infty)$ and continuous on any finite interval. Furthermore, since, as we have seen, $\int_{-\infty}^{\infty} dC_p(x)$ converges *uniformly* (with respect to p), an easy application of Helly's theorem yields

* H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, I, II, III, Mathematische Annalen, vols. 81–82 (1920), pp. 235–319, 120–64, 168–87.

† Loc. cit.

‡ Necessarily of bounded variation, by virtue of the Montel-Helly theorem extended to this class of functions.

$$\lim_{p\to\infty}\int_{-\infty}^{\infty} f(x)dC_p(x) = \int_{-\infty}^{\infty} f(x)d\psi(x), \tag{6}$$

$$\lim_{p\to\infty}\int_{-\infty}^{\infty}\frac{dC_p(x)}{z+x} = \int_{-\infty}^{\infty}\frac{d\psi(x)}{z+x} \qquad (z = iy;\ y > 0). \tag{7}$$

Consider now the expression

$$\begin{aligned} f_p(z) &\equiv \int_{-\infty}^{\infty}\frac{dC_p(x)}{z+x} = \sum_{r=0}^{\nu-1}(-1)^r\frac{\alpha_p^{(r)}}{z^{r+1}} + \frac{(-1)^\nu}{z^{\nu+1}}I_{\nu,p} \qquad (\nu \text{ arbitrary}),\\ I_{\nu,p} &= \int_{-\infty}^{\infty} x^\nu\left(\frac{z}{z+x}\right)dC_p(x),\quad \alpha_p^{(r)} = \int_{-\infty}^{\infty} x^r dC_p(x). \end{aligned} \tag{8}$$

Letting $p\to\infty$, using (7) and the property

$$\lim_{p\to\infty}\alpha_p^{(r)} = m_r \qquad (r = 0, 1, \cdots),$$

we get

$$F(z) \equiv \int_{-\infty}^{\infty}\frac{d\psi(x)}{z+x} = \sum_{r=0}^{\nu-1}\frac{(-1)^r m_r}{z^{r+1}} + \frac{(-1)^\nu}{z^{\nu+1}}\lim_{p\to\infty} I_{\nu,p}. \tag{9}$$

Observing that $|z/(z+x)|\leqq 1$ $(-\infty\leqq x\leqq\infty,\ z=iy,\ y\geqq y_0>0)$, we get

$$\begin{aligned} |I_{2s,p}| &\leqq \alpha_p^{(2s)};\\ |I_{2s-1,p}| &\leqq \left|\int_{-1}^{1} x^{2s-1}dC_p(x)\right| + \int_{1}^{\infty} x^{2s}dC_p(x) + \int_{-\infty}^{-1} x^{2s}dC_p(x)\\ &\leqq 1 + \alpha_p^{(2s)},\\ \left|\lim_{p\to\infty} I_{2s-\sigma,p}\right| &\leqq \lim_{p\to\infty}(1+\alpha_p^{2s}) = 1 + m_{2s} \qquad (\sigma = 0, 1;\ s = 1, 2, \cdots), \end{aligned}$$

$$\lim_{z=iy\to\infty}\left|z^\nu\left[F(z) - \sum_{r=0}^{\nu-1}(-1)^\nu\frac{m_r}{z^{r+1}}\right]\right| = \lim_{z=iy\to\infty}\left|\frac{1}{z}\lim_{p\to\infty} I_{\nu,p}\right| = 0. \tag{10}$$

Formula (10), where ν is arbitrary, gives the asymptotic expansion of $F(z)$:

$$F(z) \sim \sum_{r=1}^{\infty}(-1)^r\frac{m_r}{z^{r+1}} \qquad (z = iy\to\infty).$$

Hence, Hamburger's theorem is applicable and proves (γ).

Proof II. We restrict ourselves to the domain of real numbers, making use of the following extension of Helly's theorem to the infinite interval $(-\infty, \infty)$.

Given a sequence $\{v_n(x)\}$ *defined on* $(-\infty, \infty)$ *such that* (i) $v_n(x)$ *is of bounded variation on any finite interval*, (ii) *all* $v_n(x)$ *and their total variations are uniformly bounded on any finite interval*, (iii) $\lim_{n\to\infty} v_n(x) = v(x)$ *exists for all* x, *with the possible exception of a countable set of points*, (iv) $\int_a^b f(x)dv_n(x)$ *converges uniformly* (*with respect to* n) *to* $\int_{-\infty}^{\infty} f(x)dv_n(x)$ $(a\to-\infty, b\to\infty)$, *if* $f(x)$ *is continuous everywhere* (*not necessarily uniformly*). *Then* $\int_{-\infty}^{\infty} f(x)dv(x)$ *exists and* $=\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)dv_n(x)$.

We notice, first, that $v(x)$ is of bounded variation (see footnote on page 538) on any finite interval, secondly (by virtue of Helly's theorem), that

$$\left|\int_b^{b'} f(x)dv(x)\right| = \left|\lim_{n\to\infty}\int_b^{b'} f(x)dv_n(x)\right| < \epsilon$$

$(bb'>0,\ n,\ |b|,\ |b'|$ sufficiently large; $\epsilon>0$ arbitrarily small); hence, $\int_{-\infty}^{\infty} f(x)dv(x)$ *exists*. Furthermore,

$$\Delta_n = \left|\int_{-\infty}^{\infty} fdv - \int_{-\infty}^{\infty} fdv_n\right| \leqq \left|\int_{-\infty}^{a} fdv\right| + \left|\int_{-\infty}^{a} fdv_n\right| + \left|\int_{b}^{\infty} fdv\right|$$
$$+ \left|\int_{b}^{\infty} fdv_n\right| + \left|\int_{a}^{b} fdv - \int_{a}^{b} fdv_n\right| \qquad (a<0,\ b>0)$$

can be made as small as we please by taking $-a$, b, and then n sufficiently large. Hence $\lim_{n\to\infty}\Delta_n = 0$.

Remark. *The new condition* (iv) *is essential.* The following example shows that *if* (iv) *is not satisfied, the theorem may not hold*: $f(x)=x$, $v_n(x)=0$ $(x\leqq 0)$, $1-1/2^n (0<x\leqq 4^n)$, $1 (x>4^n)$. Here

$$\lim_{n\to\infty} v_n(x) = v(x) = 0\ (x \leqq 0),\ 1(x>0);$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} xdv_n(x) = \lim_{n\to\infty} 4^n\cdot\frac{1}{2^n} = \infty \neq \int_{-\infty}^{\infty} xdv(x) = 0.^*$$

It suffices to apply the above theorem, with $f(x)=x^r (r=0, 1, \cdots)$, to the above sequence $\{C_p(x)\}$ which satisfies *all four conditions* stated, and (γ) is established.

6. Special case. A direct corollary is the following

THEOREM. *If* $\lim_{n\to\infty} m_n^r$ *exists* $(=m_r)$ *for* $r=0, 1, \cdots$, *then at least one fixed law of probability, say* $F(x)$, *exists such that* m_r *is its rth moment* $(r=0, 1, \cdots)$, *and a subsequence* $\{C_i(x)\equiv F_{n_i}(x)\}$ *can be extracted from the given*

* $v_n(x)$, $v(x)$ have each a single saltus $=1/2^n$, 1 at $x=4^n$, 0 respectively.

sequence $\{F_n(x)\}$ *of laws of probability such that* $\lim_{i\to\infty} C_i(x) = F(x)$ *for any* x. *If, in addition, the* $\{m_r\}$ *are such that the corresponding moments-problem is determined, then the sequence* $\{F_n(x)\}$ *itself converges, for* $n\to\infty$, *to* $F(x)$ *at any point of continuity of* $F(x)$.

We need a proof for the last part only. Assume that a point x_0 of continuity of $F(x)$ exists such that $\{F_n(x_0)\}$ does not converge to $F(x_0)$. Hence, a subsequence $\{C_k(x_0) \equiv F_{n_k}(x_0)\}$ can be extracted such that $C_k(x_0)$ converges, for $k\to\infty$, to a certain number $A \neq F(x_0)$. On the other hand, we have seen that the sequence $\{C_k(x)\}$ gives rise to a subsequence $\{d_i(x)\}$ which, *for any* x, converges, as $i\to\infty$, to a function $d(x)$, *having the same moments* $m_r (r=0, 1, \cdots)$ *as* $F(x)$, and therefore, since, by hypothesis, the moments-problem corresponding to $\{m_r\}$ is *determined*,

$$\lim_{i\to\infty} d_i(x_0) = d(x_0) = F(x_0) \tag{11}$$

($F(x)$ being continuous at $x=x_0$), which is impossible, $\{d_i(x_0)\}$ being a subsequence of $\{C_k(x_0)\}$ which *converges, but not to* $d(x_0)$. We have seen also ((i), p. 537) that $F(-\infty)=0$, $F(\infty)=1$.

The condition that the moments-problem corresponding to $\{m_r\}$ be determined is not only sufficient for the validity of the limiting relation

$$\lim_{n\to\infty} F_n(x) = F(x)$$

(at any point of continuity of $F(x)$),* but *it is also necessary*. For if $F(x)$ and $\phi(x)$ be *two distinct* solutions of the moments-problem in question, then $F_n(x)$ should converge simultaneously to $F(x)$ and $\phi(x)$ at all points of continuity, while at least one such point x_0 exists where $F(x_0) \neq \phi(x_0)$.†

7. **The classical case:** $m_r = \pi^{-1/2}\int_{-\infty}^{\infty} x^r e^{-x^2} dx (r=0, 1, \cdots)$. Here

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^r dF_n(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^r e^{-x^2} dx \quad (r = 0, 1, 2, \cdots)$$

implies

$$\lim_{n\to\infty} \int_{-\infty}^{x} dF_n(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-x^2} dx \ (x \text{ arbitrary}).$$

* Even everywhere in $(-\infty, \infty)$, $F(x)$ being a law of probability, hence continuous to the left (see Introduction).

† The conditions imposed by different writers on the quantities $\{m_r\}$ are such as *to ensure the determined character of the corresponding moments-problem*. In fact, one sees readily that the conditions of R. von Mises, Pólya, and Cantelli (loc. cit.)

$$m_{2n} \leqq C\left(\frac{n}{c^2 e}\right)^n (C, c = \text{const.}), \ \lim_{n\to\infty} m_{2n}^{1/(2n)}/n \text{ is finite}, \ m_{2n}\frac{[\psi(2n)]^{2n}}{2n!} < 1 \ (n > n_1;\ \psi(n)_{n\to\infty} \to \infty)$$

are but special cases of Carleman's condition (2).

In fact, the moments-problem corresponding to $\{m_r\}$ is determined (by virtue of (2) or (3)), and

$$F(x) = \pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$$

is continuous for all x.

We see that *the classical case is but a very special case of the general second limit-theorem.*

8. **The determined character of the moments-problem in the classical case.** The conditions (2, 3) ensuring the determined character of the moments-problem have been established by means of very profound, but also complicated, considerations (continued fractions, singular integral equations). It seems of interest to give an *elementary* proof involving a simple theorem of Pólya.*

We wish to prove the following

THEOREM. *Given a law of probability $\psi_1(x)$ such that*

$$(12)\quad \begin{aligned} m_{2r} &= \int_{-\infty}^{\infty} x^{2r}d\psi_1(x) = \Gamma\left(\frac{2r+1}{\lambda}\right)\Big/\Gamma\left(\frac{1}{\lambda}\right) = \left(\frac{\lambda}{2\Gamma(1/\lambda)}\right)\int_{-\infty}^{\infty} x^{2r}e^{-|x|^{\lambda}}dx, \\ m_{2r+1} &= \int_{-\infty}^{\infty} x^{2r+1}d\psi_1(x) = 0 = \frac{\lambda}{2\Gamma(1/\lambda)}\int_{-\infty}^{\infty} x^{2r+1}e^{-|x|^{\lambda}}dx \end{aligned}$$

$$(\lambda \geqq 1;\ r = 0, 1, \cdots).$$

Then necessarily

$$\psi_1(x) = \frac{\lambda}{2\Gamma(1/\lambda)}\int_{-\infty}^{x} e^{-|x|^{\lambda}}dx$$

for any x. In other words, *$\psi_1(x)$ is uniquely determined by* (12).

Assume the existence of two such functions $\psi_1(x)$ and $\psi_2(x)$. Employing the reasoning of §1 (footnote on page 534) and using property (iii), page 537.

$$\lim_{x\to\infty} x^s(1-\psi_i(x)) = \lim_{x\to-\infty} |x|^s\psi_i(x) = 0 \quad (i = 1, 2;\ s>0 \text{ arbitrary}),$$

we conclude that

$$(13)\quad \int_{-\infty}^{\infty} x^l F(x)dx = 0 \quad (l = 0, 1, \cdots;\ F = \psi_1 - \psi_2 = (1-\psi_2) - (1-\psi_1)).$$

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* G. Pólya, *Über den Gaussschen Fehlergesetz,* Astronomische Nachrichten, vol. 208 (1919), No. 4981, pp. 185–192.

On the other hand, a reasoning similar to that of §4 leads, making use of (12) and the asymptotic expression for the Γ function, to

$$(14)\qquad \begin{aligned} &|F(x_0)| < \frac{2m_{2n}}{x_0^{2n}} && (n \text{ very large, } x_0 \text{ arbitrary}),\\ &|F(x_0)| < Ce^{(-1/2)|x_0|^\lambda} && \left(\left(\frac{2n}{\lambda}\right)^{2/\lambda+1/(2n)} < x_0^2 < \left(\frac{4n}{\lambda}\right)^{2/\lambda}\right),\\ & && C=\text{const. independent of } n \text{ and } x_0. \end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} |F(x)|\, e^{c|x|} dx \qquad (0 < c < \tfrac{1}{2})$$

exists. But this is precisely the condition imposed by Pólya,* which leads to the required conclusion: $F(x) \equiv 0$ *at all points of continuity*.†

* Loc. cit. p. 187. For the classical case ($\lambda = 2$) cf. M. H. Stone, *Developments on Hermite polynomials*, Annals of Mathematics, vol. 29 (1927), pp. 1–13.

† After the present paper had been prepared for publication, we came across an interesting article by A. Wintner: *Über den Konvergenzsatz der mathematischen Statistik*, Mathematische Zeitschrift, vol. 28 (1928), pp, 470–480, some of the results of which are similar to those obtained above.

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